Controlling the spreading in small-world networks

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Abstract

The spreading (propagation) of diseases, viruses, and disasters such as power blackout through a huge-scale and complex network is one of the most concerned issues today. In this paper, we study the control of such spreading in a nonlinear spreading model of small-world networks. We found that the short-cut adding probability p in the N-W model [7] of small-world networks determines the Hopf bifurcation and other bifurcating behaviors in the proposed model. We further show a control technique that stabilize a periodic spreading behavior onto a stable equilibrium over the proposed model of small-world networks.

Keywords: Spreading dynamics, Hopf bifurcation, small-world network

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1 Introduction

Recently, Watts and Strogatz proposed a small-world model [11], named W-S model, which consists of a rewired regular lattice with a very small fraction of long-range connections. Newman and Watts [7] then made a slight modification on the rewiring process of the W-S model, named N-W model, by means of adding linkages between pairs of randomly chosen nodes with a very small probability $0 < p \ll 1$. When selecting a very small probability $0 < p \ll 1$ in these models, the obtained small-world networks have large clustering and small average distances, which match the familiar small-world feature discovered in many real-life large-scale networks [3, 6, 8, 9, 10].

Since the discovery of the small-world phenomenon, many interesting results on the analysis, control

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and applications of small-world networks have been published [5, 9, 10, 12]. In particular, Moukarzel [5] studied a linear disease spreading model of small-world networks, which was further extended to a nonlinear model with frictions and time-delays where chaos and bifurcation can emerge [2, 12]. In this paper, we concern with the Hopf bifurcation phenomenon during the spreading process in small-world networks, and its control problem, by utilizing the short-cut adding probability p as the means to stabilize the bifurcating behaviors.

The remainder of this paper is organized as follows: Section 2 describes the nonlinear spreading model, where the existence of Hopf bifurcation in this model is discussed. Section 3 proposes a method of of controlling the bifurcating behaviors using the short-cut adding probability p. The effectiveness of this control technique is visualized via a numerical example, also in this section. Finally, Section 4 concludes the paper.

2 Nonlinear spreading model and Hopf bifurcation

Assume that a disease (virus, power blackout) spreads with a constant radial velocity $v = 1$ from an original infection site of a network. The infected volume $V(t)$ grows according to the following nonlinear differential equation:

$$\frac{dV(t)}{dt} = 1 + 2pV(t-\delta) - \mu(1+2p)V^2(t-\delta) \tag{1}$$

where δ is the time-delay during the spreading. The nonlinear friction term consists of two parts: the former comes from the nonlinear interaction within the regular lattice, while the latter is the nonlinear interaction coming from the newly added links, which is dependent on the probability p in the N-W small-world model [2]. And $\mu > 0$ is a measure of such nonlinear frictions.

It should be noted that Eq. (1) contains as special cases all the existing linear models of Newman and Watts [7] and Moukarzel [5]. When $\mu = 0$ and $\delta = 0$, obviously the model produces exponential growth without limit. Besides, the nonlinear model of Yang [12] hides the effect of the probability p on bifurcation behaviors in spreading, while our model (1) is more general and uncovers this effect. Therefore, model (1) is more suitable for studying the spreading phenomenon in small-world networks.

Denote $s(t) = V(t) - V^*$, where V^* is the equilibrium of Eq. (1) with $V(t) > 0$ and is given by

$$V^* = \frac{p + \sqrt{p^2 + \mu(1+2p)}}{\mu(1+2p)} \tag{2}$$

To study the stability of the spreading behaviors in model (1), we select μ as the parameter of bifurcation for analysis. We have the following theoretical results for the existence of Hopf bifurcation in model (1), including the directions, stabilities and periods of the bifurcating periodic solutions.

Theorem 1 *If $\delta < \frac{\pi}{4p}$, then when the positive parameter μ passes through the critical value $\mu_0 = \frac{\pi^2 - 16\delta^2 p^2}{16\delta^2(1+2p)}$, there is a Hopf bifurcation in model (1) at its equilibrium V^*.*

Proof: See reference [2].

To state the next theorem, define

$$\overline{B} = \frac{1}{1 - 2\sqrt{p^2 + \mu(1+2p)}\,\delta e^{-i\omega_0\delta}}, \qquad g_{20} = 2\mu(1+2p)\overline{B},$$

$$g_{11} = -2\mu(1+2p)\overline{B}, \qquad g_{02} = 2\mu(1+2p)\overline{B},$$

$$g_{21} = -2\mu(1+2p)\overline{B}i\left[-2\frac{g_{11} + \overline{g}_{11} + 2\mu(1+2p)}{2\sqrt{p^2 + \mu(1+2p)}} + \frac{-g_{20} - \overline{g}_{02} + 2\mu(1+2p)}{2i\omega_0 + 2\sqrt{p^2 + \mu(1+2p)}}\right],$$

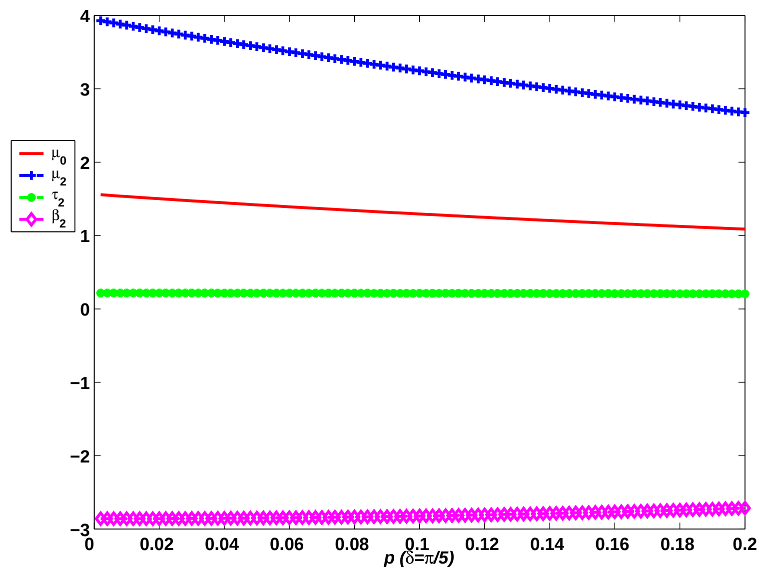

Figure 1: The evolution trend of bifurcation parameters $\mu_0(p)$, $\mu_2(p)$, $\tau_2(p)$, $\beta_2(p)$, with $0 < p \leq 0.2$ in model (1) (small-world networks), where $\delta = \frac{\pi}{5}$.

$$C_1(0) = \frac{i}{2\omega_0}\left[g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{02}\right|^2\right] + \frac{g_{21}}{2},$$

$$\alpha'(0) = \frac{d}{d\mu}\left[\text{Re }\lambda\right]\Big|_{\mu=\mu_0} = \frac{8(1+2p)\delta}{4+\pi^2}, \qquad \omega'(0) = \frac{d}{d\mu}\left[\text{Im }\lambda\right]\Big|_{\mu=\mu_0} = \frac{16\delta(1+2p)}{\pi(4+\pi^2)},$$

$$\mu_2 = -\frac{\text{Re }C_1(0)}{\alpha'(0)}, \qquad \tau_2 = \frac{-\text{Im }C_1(0) + \mu_2\omega'(0)}{\omega_0}, \qquad \beta_2 = 2\text{Re }C_1(0).$$

Theorem 2 *Parameter μ_2 determines the direction of the Hopf bifurcation: If $\mu_2 > 0$ (< 0), the Hopf bifurcation is supercritical (subcritical) and bifurcating periodic solutions exist for $\mu > \mu_0$ $(< \mu_0)$; parameter β_2 determines the stability of the bifurcating periodic solutions: the solutions are orbitally stable (unstable) if $\beta_2 < 0$ (> 0); and parameter τ_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $\tau_2 > 0$ (< 0).*

Proof: See reference [2].

3 Control of bifurcating behaviors in small-world networks

As can be seen from Theorems 1-2, the critical value of μ_0 for the occurrence of Hopf bifurcation, and those values of μ_2, τ_2, β_2 for verifying the stability of bifurcating periodic solutions, are all dependent on the probability p, which will be denoted by $\mu_0(p)$, $\mu_2(p)$, $\tau_2(p)$, $\beta_2(p)$ below.

Generally, small-world networks are described by W-S/N-W models [7, 11] with a small probability $0 < p \ll 1$. Figure 1 shows the values of $\mu_0(p)$, $\mu_2(p)$, $\tau_2(p)$, $\beta_2(p)$ when $0 < p \leq 0.2$, where, interestingly, we have found that $\tau_2(p)$ and $\beta_2(p)$ are not sensitive to the change of the probability p. It implies that these small-world networks have almost the same bifurcating periodic behaviors, for example, $\tau_2(0.01) = 0.2162$, which is almost the same as $\tau_2(0.1) = 0.2135$. This means that when μ increases after passing through the critical value $\mu_0(p = 0.01, 0.1)$, the periods of these periodic solutions increase in about the same amount.

Recall that bifurcation control deals with a modification of some bifurcation characteristics of a parameterized nonlinear system by a judiciously designed control input [1]. Because p determines the

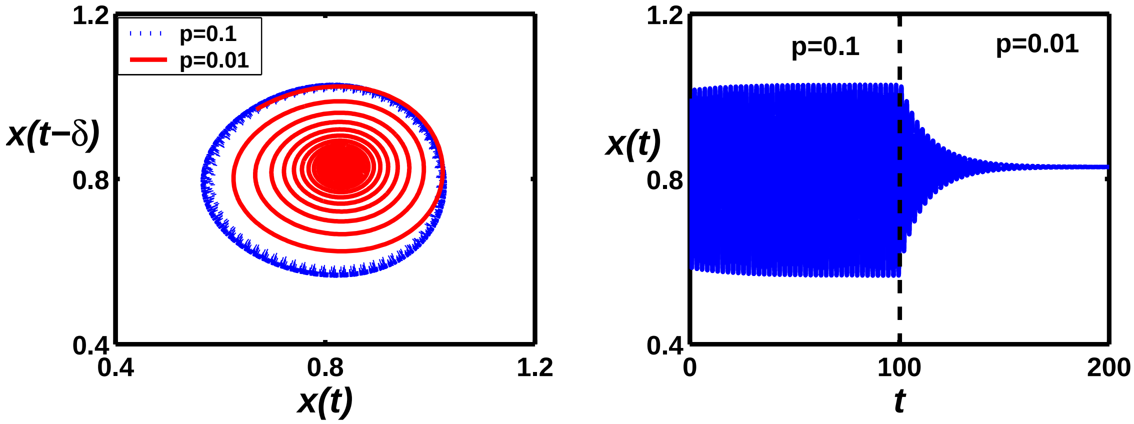

Figure 2: Phase plot and waveform plot of a small-world network of model (1), with $p = 0.1$ (dotted line), and $p = 0.01$ (solid line), both $\delta = \frac{\pi}{5}$ and $\mu = 1.45$ (p changes at $t = 100$).

bifurcating values of $\mu_0(p)$, $\mu_2(p)$, which are monotonically decreasing as p increases with a fixed time-delay δ. Therefore, if we want to apply a controller to the bifurcating behaviors, p is preferred to be controlled. We thus employ the following control strategy:

$$\frac{dV(t)}{dt} = 1 + 2u(p)V(t-\delta) - \mu(1 + 2u(p))V^2(t-\delta), \tag{3}$$

where

$$u(p) = p - \Delta p. \tag{4}$$

We may select p as the control parameter to first calculate the bifurcation parameter $\mu_0(p)$, and then to vary p with Δp so as to obtain a different $\mu_0(p)$. This results in different bifurcating spreading behaviors in a small-world network. To this end, we can further stabilize periodic spreading solutions onto desired stable equilibria, and vice versa.

Figure 2 shows an example. At the beginning, $t \in [0, 100]$, we set $p = 0.1$, $\delta = \frac{\pi}{5}$, $\mu = 1.45$ in model (1), and then compute $\mu_0(0.1) = 1.2938 < \mu = 1.45$. As expected, a Hopf bifurcation occurs and the spreading behavior is a periodic solution. To stabilize this periodic orbit onto a stable equilibrium, we vary the probability p to be 0.01 after $t = 100$. Since $\mu_0(0.01) = 1.5315 > \mu = 1.45$, the periodic spreading solution asymptotically converges to the stable equilibrium $V^* = 0.8291$, as shown in Fig. 2.

4 Conclusion and discussion

In this paper, we have introduced a general nonlinear spreading model of the small-world network type, which has a flexible nonlinear interaction effect. Based on the study of Hopf bifurcation in this spreading model, a simple bifurcation control example has been simulated, showing that the probability p can be used as a control parameter to stabilize a periodic spreading behavior onto a stable equilibrium in the small-world network.

More recently, Motter [4] studied a means of controlling cascade failure in a complex network, and pointed out that selective removal of network elements can prevent the cascade from propagating through the entire network. Such a selective removal control strategy is actually a complementary approach of the technique discussed in this paper, for controlling the spreading in small-world networks.

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